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# Extrapolating the mean-values of multiplicative functions

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ABSTRACT

It is shown that certain commonly occurring conditions may be factored out of sums of multiplicative arithmetic functions.

A function is *arithmetic* if it is defined on the positive integers. Those complex-valued arithmetic functions $g$ which satisfy the relation $g(ab)=g(a)g(b)$ for all coprime pairs of positive integers $a, b$ are here called *multiplicative*. In this paper $g$ will be a multiplicative function which satisfies $|g(n)|\le 1$ for all positive integers $n$.

THEOREM 1. *Let $1\le w_0\le x$. There is a real $\tau$, $|\tau|\le(\log x)^{1/19}$, so that*

$$\sum_{n\le x/w} g(n)=w^{-1+i\tau}\sum_{n\le x} g(n)+O\left(\frac{x}{w}\left(\frac{\log 2w_0}{\log x}\right)^{1/19}\right)$$

*uniformly for $1\le w\le w_0$. If $g$ is real-valued, then we may set $\tau=0$. The implied constant is absolute.*

THEOREM 2. *There is a real $\tau$, $|\tau|\le(\log x)^{1/19}$, so that*

$$\sum_{\substack{n\le x\\ (n,D)=1}} g(n)=\eta(D)\sum_{n\le x} g(n)+O\left(\frac{x(\log\log 3D)^2}{(\log x)^{1/19}}\right),$$

*with*

$$\eta(D)=\prod_{p|D}\Big(1+\sum_{k\le\log x/\log p} p^{-k(1+i\tau)}g(p^k)\Big)^{-1},$$

*holds uniformly for $x \geq 2$ and odd integers D. It holds for even integers as well, provided*

$$|1 + \sum_{k \leq \log x/\log 2} 2^{-k(1+i\tau)} g(2^k)| \geq c_1 > 0.$$

*In this case $\tau$ is determined for the odd factor of D, and the error term depends upon $c_1$. Otherwise the implied constant is again absolute. For real-valued g we may set $\tau = 0$.*

These theorems show that conditions which are not too severe may be factored out of sums of multiplicative functions.

It follows from Theorem 1 that

$$(1) \qquad |\sum_{n \leq x/w} g(n)| = \frac{1}{w} |\sum_{n \leq x} g(n)| + O\left(\frac{x}{w}\left(\frac{\log 2w}{\log x}\right)^{1/19}\right),$$

where we may remove the moduli if $g$ is real valued. This improves an estimate of Hildebrand [11], derived by a different method.

As an application, let $\chi$ be a non-principal character of order $m \geq 2$, defined with respect to a prime modulus $p$. Let $\varrho$ be a value which $\chi$ can assume, and let $N(\varrho)$ denote the least positive integer $t$ for which $\chi(t) = \varrho$.

Employing the estimate (1) with $g = \chi^j$, $j = 1, \ldots, m$ in turn, and setting $x = p^{1/4+\delta}$ so that we may estimate the sums over the range $1 \leq n \leq x$ by the well-known character sum bound of Burgess [1], [2], we may choose $w$ to obtain $N(\varrho) \ll p^{\alpha}$ with $\alpha$ of the form $1/4 - c_2 m^{-19}$, for a positive absolute constant $c_2$. The result of Hildebrand gives for $\alpha$ a value $1/4 - \exp(-c_3 m^{-2})$.

It should be remarked that for questions of this type the method of Elliott [3], [4] Chapter 4, pp. 155–158, is much the most elementary. It gives for $\alpha$ at once $1/4 - \exp(-c_4 m^{-3})$ and, with straightforward changes, essentially the same bound as the method of Hildebrand.

Since the exponent 1/19 in Theorems 1 and 2 could be reduced somewhat, a corresponding improvement in the saving $c_2 m^{-19}$ could be obtained.

Theorem 1 may be combined with the following result derived from the dual of the Turán-Kubilius inequality

$$\sum_{p \leq x} \frac{1}{p} |g(p) \frac{p}{x} \sum_{n \leq x/p} g(n) - \frac{1}{x} \sum_{n \leq x} g(n)|^2 \ll 1.$$

See, for example, Elliott [4], Chapter 4, Lemma (4.7), p. 147, where we set $a_n = g(n)$. In particular, for real-valued functions $g$,

$$x^{-1} \sum_{n \leq x} g(n) \ll \left(\sum_{p \leq x} \frac{1}{p} |1 - g(p)|^2\right)^{-1/2}.$$

This may be compared with a similar result of Hildebrand, [10], obtained by a different method. For functions essentially bounded away from zero better can be done (Halász [9], an account of which is given in Elliott [5], Chapter 19), but at the expense of a much more complicated proof.

If the function $g$ is completely multiplicative, then $\eta(D)$ in Theorem 2 may be replaced by

$$\prod_{p|D} (1-p^{-(1+i\tau)}g(p))^{-1}.$$

In particular, Theorem 2 is useful in reducing sums involving characters to sums involving primitive characters.

If $g^m=1$, or $g$ is a character of order not exceeding $m$, then Theorems 1 and 2 in fact hold with $\tau=0$ provided the exponent $1/19$ be replaced with another of the form $c_5 m^{-3}$. This may be proved by combining Lemma (19.6) of Elliott, [5] with the method of the present paper.

Let $M$ denote the set of integers not exceeding $x$ which are not divisible by any prime $p \leq \exp((\log x)^\gamma)$. Let $M(x)$ denote its cardinality. Then without employing any analytic continuation of the Riemann zeta function into the half-plane Re $(s)\leq 1$, we obtain for the Möbius function $\mu$ the estimate

$$M(x)^{-1} \sum_{n\leq x, n\in M} \mu(n) \ll (\log\log x)^{-1/2}+(\log x)^{3\gamma-1/19}.$$

For other applications of versions of Theorems 1, 2, to the study of differences $f_1(an+b)-f_2(An+B)$ of additive arithmetic functions, see Elliott [7].

Let

$$G(s)=\sum_{n=1}^{\infty} g(n)n^{-s}$$

with $s$ complex, $\sigma=\text{Re}(s)>1$, be the Dirichlet series associated with $g$. $\zeta(s)$ will denote the Riemann Zeta function, given by $g(n)=1$.

The proofs of the theorems employ an integral transform. For real positive $y$

$$\frac{1}{2\pi i}\int_{(\alpha)} \frac{y^s}{s^2}\,ds=\begin{cases} 0 & \text{if } 0<y\leq 1,\\ \log y & \text{if } y>1,\end{cases}$$

the integration being taken over the line Re $(s)=\alpha>0$, from $-i\infty$ to $i\infty$. For $\alpha>1$ we obtain a representation

$$S(y)=y^{-1}\sum_{n\leq y} g(n)\log n \log\frac{y}{n}=-\frac{1}{2\pi i}\int_{(\alpha)}\frac{G'(s)y^{s-1}}{s^2}\,ds$$

by inverting the order of integration and summation.

I estimate this last integral by modifying a method of Halász, an account of which is given in Elliott [4], Chapter 6, pp. 233–241. The procedure is to choose positive parameters $K\geq 2$, $M\geq 2$, and treat separately the three ranges of integration $|\tau|\leq K(\alpha-1)$, $K(\alpha-1)<|\tau|\leq M$, $|\tau|>M$. From that account I employ the following results.

LEMMA 1. *For* $1<\alpha\leq 2$, $N\geq 0$,

$$\frac{1}{2\pi}\int_{\substack{(\alpha)\\ |\tau|>N}} \left|\frac{G'(s)}{sG(s)}\right|^2 d\tau<\frac{e^{42}}{(N+1)(\alpha-1)}.$$

*Moreover, if $\delta > 0$,*

$$\int_{(\alpha)} \frac{|G(s)|^{1+\delta}}{|s|^2}\, d\tau \ll \frac{1}{(\alpha-1)^{\delta}},$$

*the implied constant depending upon $\delta$.*

PROOF. On pages 234–236 of Elliott [4], Chapter 6, it is shown that for $1 < \alpha \le 2$

$$(2) \qquad \frac{1}{2\pi} \int_{(\alpha)} \left|\frac{G'(s)}{sG(s)}\right|^2 d\tau < \frac{e^{36}}{(\alpha-1)}.$$

In particular this holds if the range of the integral is curtailed to $|\tau| \le 1$. Hence

$$\frac{1}{2\pi} \int_{\substack{(\alpha)\\ |\tau| \le 1}} \left|\frac{G'(s)}{G(s)}\right|^2 d\tau < \frac{9e^{36}}{\alpha-1}, \quad 1 < \alpha \le 2.$$

This inequality continues to hold if $g(n)$ is everywhere replaced by $g(n)n^{im}$ for an integer $m$. Equivalently

$$\frac{1}{2\pi} \int_{\substack{(\alpha)\\ |\tau-m| \le 1}} \left|\frac{G'(s)}{G(s)}\right|^2 d\tau < \frac{9e^{36}}{\alpha-1}, \quad 1 < \alpha \le 2.$$

The first of the integrals to be estimated in the lemma does not exceed

$$\sum_{|m|>N-1} \frac{1}{2\pi} \int_{\substack{(\alpha)\\ |\tau-m| \le 1}} \left|\frac{G'(s)}{sG(s)}\right|^2 d\tau < \frac{9e^{36}}{\alpha-1} \sum_{|m|>N-1} \frac{1}{m^2},$$

and the asserted result follows readily.

For $\delta = 1/2$ a detailed proof of the second inequality of the lemma is carried out in Elliott [4], Chapter 6, pp. 237-238. The same method gives the present result.

REMARK. In an appendix to the present paper I give a short proof that for functionals of the type estimated in Lemma 1, the Riemann zeta function is essentially an extremal. From this one may readily deduce the above bounds.

The following result is essential.

LEMMA 2. *The inequality*

$$|G(\sigma+i\tau_1)|\,|G(\sigma+i\tau_2)| \le e^{28}(\zeta(\sigma)^3|\zeta(\sigma+i(\tau_1-\tau_2))|)^{1/2}$$

*holds uniformly for $\sigma > 1$ and real $\tau_j$.*

PROOF. This is a particular case of Lemma 2 of Elliott [6].

It is convenient to begin by establishing Theorem 1 for real-valued functions.

PROOF OF THEOREM 1 FOR REAL-VALUED $g$

Let $1 \leq y \leq x$, $x \geq e^4$ hold. We apply the above integral representation for $S(y)$, with $\alpha = 1 + (\log x)^{-1}$.

Employing the decomposition $G' = (G'/G)G$ and applying the Cauchy-Schwarz inequality we obtain

(3) $$\int_{\substack{(\alpha)\\ |\tau|>M}} \left|\frac{G'(s)}{s^2}\right| d\tau \leq \left( \int_{\substack{(\alpha)\\ |\tau|>M}} \left|\frac{G'(s)}{sG(s)}\right|^2 d\tau \int_{\substack{(\alpha)\\ |\tau|>M}} \left|\frac{G(s)}{s}\right|^2 d\tau \right)^{1/2}.$$

From Lemma 1 with $\delta = 1$ this is clearly $\ll (M+1)^{-1/2}(\alpha-1)^{-1}$, and indeed better could be done if it were necessary. Setting $M = (\log x)^{1/2}$ we obtain towards $S(y)$ a contribution of $O((\log x)^{3/4})$. Note that because of our restrictions upon the size of $y$, $|y^{s-1}| \leq x^{\alpha-1} = e$. The large values of $\tau$ do not contribute significantly to the integral.

On the line-segments $K(\alpha-1) < |\tau| \leq (\log x)^{1/2}$ we apply the bound

$$|G(\alpha + i\tau)| \leq e^{14}(\zeta(\alpha)^3 |\zeta(\alpha + 2i\tau)|)^{1/4}$$

guaranteed by Lemma 2. The well-known estimate

$$\zeta(s) \ll \frac{1}{|s-1|} + \log(2 + |s|), \quad \sigma > 1,$$

derived, for example, in Elliott [4], Chapter 2, Lemma (2.14), p. 98, shows that on these segments.

(4) $$G(s)\zeta(\alpha)^{-1} \ll \left(\frac{1}{K} + \frac{\log\log x}{\log x}\right)^{1/4}$$

holds uniformly.

Let $0 \leq \lambda < 1$. We again apply the Cauchy-Schwarz inequality, as at (3), this time with the range(s) of integration $J: K(\alpha-1) < |\tau| \leq (\log x)^{1/2}$. The second of the integrals in the upper bound is estimated by

$$\max_{s \in J} |G(s)|^{\lambda} \int_J \frac{|G(s)|^{2-\lambda}}{|s|^2} d\tau.$$

This we treat in turn by (4), and Lemma 1 with $\delta = 1 - \lambda$. Continuing with another application of Lemma 1, this time with $N = 0$, we see that the middle range of $\tau$ contributes

$$\ll \left(\frac{1}{K} + \frac{\log\log x}{\log x}\right)^{\lambda/8} \log x$$

towards $S(y)$.

We replace $y$ by $x/w$, and on the interval $|\tau| \leq K(\alpha-1)$ estimate $w^{-(s-1)}$ by $1 + O((1+K)\log w/\log x)$. Since by (3) with $M = 0$, and Lemma 1 we have

$$\int_{\substack{(\alpha)\\ |\tau| \leq K(\alpha-1)}} \frac{|G'(s)|}{|s|^2} d\tau \ll \log x,$$

we reach

$$S\left(\frac{x}{w}\right)=-\frac{1}{2\pi i}\int_{\substack{(\alpha)\\|\tau|\le K(\alpha-1)}}\frac{G'(s)x^{s-1}}{s^2}\,ds+O\left(\left\{\frac{1}{K^{\lambda/8}}+\frac{(1+K)\log w}{\log x}+\right.\right.$$
$$\left.\left.+\left(\frac{\log\log x}{\log x}\right)^{\lambda/8}\right\}\log x\right).$$

Note that the integral in this representation does not depend upon $w$.

Set $K^{\lambda/8+1}=\log x/\log 2w_0$, and apply this estimate twice, once with $w=1$. Subtraction gives

$$S\left(\frac{x}{w}\right)=S(x)+O\left(\left(\frac{\log 2w_0}{\log x}\right)^{\lambda/(8+\lambda)}\log x\right)$$

uniformly for $1\le w\le w_0\le x$, $x\ge e^4$.

To remove the weights $\log x/n$, $\log x/wn$ which occur in the definitions of $S(x)$, $S(x/w)$ respectively, consider

$$\frac{z}{\varepsilon}\,(S(z)-(1-\varepsilon)S(z(1-\varepsilon)))=\sum_{n\le z}g(n)\log n+O(\varepsilon z\log z+\log z),$$

which holds uniformly for $0<\varepsilon\le 1/2$, $z\ge 1$. Employing this for $z=x$, $z=x/w$ in turn, and subtracting, choosing $\varepsilon=(\log 2w_0/\log x)^{\varrho}$, $\varrho=\lambda(16+2\lambda)^{-1}$, we obtain

$$\frac{w}{x}\sum_{n\le x/w}g(n)\log n=\frac{1}{x}\sum_{n\le x}g(n)\log n+O\left(\left(\frac{\log 2w_0}{\log x}\right)^{\varrho}\log x\right),$$

certainly if $(\log 2w_0/\log x)^{\varrho}\le 1/2$. However, if this last fails, then the assertion is trivially valid.

The weights $\log n$ may be removed by employing the estimate

$$\Big|\sum_{n\le t}g(n)\log(t/n)\Big|\le\sum_{n\le t}\log(t/n)\ll t,\quad t\ge 1,$$

for $t=x/w$, $x$ in turn, and dividing by $\log x$. Since any $\lambda<1$ may be chosen, every $\varrho<1/18$ is permissible. This completes the proof of Theorem 1 for real-valued functions $g$.

REMARK. Employing the expansion

$$w^{-(s-1)}=\sum_{j=0}^{k-1}\frac{1}{j!}\,(-(s-1)\log w)^j+$$
$$+O\left((|s-1|\log w)^k\exp\left((K+1)\frac{\log w}{\log x}\right)\right),$$

we can obtain an asymptotic expansion of the form

$$\frac{w}{x}\sum_{n\le x/w}g(n)=\frac{1}{x}\sum_{n\le x}g(n)+\sum_{j=0}^{k-1}\phi_j(\log w)^j+E,\quad 1\le w\le w_0\le x.$$

Here each $\phi_j$ is a function of $x$, $w_0$, and for any fixed $\varepsilon>0$ a bound $E\ll \ll((\log 2w_0)/\log x)^{1/16-\varepsilon}$ may be arranged by taking $k$ sufficiently large in terms of $\varepsilon$ only.

LEMMA 3. *Define*

$$h(2)=\sum_{k=1}^{\infty} g(2^k)2^{-ks}.$$

*Then there is a representation*

$$G(s)=(1+h(2))\exp\,(\sum_{p\geq 3} g(p)p^{-s})G_1(s)$$

*with* $e^{-5}\leq |G_1(s)|\leq e^5$, *valid in the half-plane* $\sigma>1$.

PROOF. This assertion is contained in Lemma (6.6), p. 230 of [4], derived from the Euler product representation for $G(s)$.

LEMMA 4. *Let* $\sigma>1$, $0\leq T_0\leq T$. *Then there is a real* $\tau$, $|\tau|\leq T$, *such that*

$$\zeta(\sigma)^{-1}|G(\sigma+i(\tau+t))|\leq e^{14}\max_{T_0\leq |t|\leq T}(\zeta(\sigma)^{-1}|\zeta(\sigma+it)|)^{1/4} \tag{5}$$

*uniformly for* $T_0\leq |t|\leq T$. *Moreover, either* $\tau=0$, *or* $|\tau|\geq T_0$ *and with* $\beta$ *denoting this upper bound*

$$\sum_p \frac{1}{p^\sigma}(1-\mathrm{Re}\; g(p)p^{-i\tau})\leq \log\frac{1}{\beta}+12. \tag{6}$$

PROOF. If $|G(\sigma+it)|\leq\beta$ for $T_0\leq |t|\leq T$, then the main inequality (5) is satisfied with $\tau=0$. I shall therefore assume that $|G(\sigma+i\tau)|>\beta$ for some $\tau$ in the same range. The bound (5) now follows from Lemma 2 with $\tau_1=\tau+t$, $\tau_2=\tau$. Moreover, if we cannot take $\tau=0$, then applying Lemma 3 to both $G(s)$ and $\zeta(s)$,

$$\beta\leq\zeta(\sigma)^{-1}|G(\sigma+i\tau)|\leq 2e^{10}\exp\,(-\sum_{p\geq 3} p^{-\sigma}(1-\mathrm{Re}\; g(p)p^{-i\tau})).$$

Inequality (6) is immediate.

LEMMA 5. *Let $b$ be a real number,*

$$L=\sum_{p\leq x}\frac{|g(p)-1|}{p}.$$

*Then*

$$\sum_{n\leq x} g(n)n^{ib}=\frac{x^{ib}}{1+ib}\sum_{n\leq x} g(n)+O(xe^{L/2}(\log x)^{-1/2}(\log(|b|+\log x))^2).$$

PROOF. This is Lemma (8.6) of Ruzsa [12]. It is established by the method of Halász [8].

PROOF OF THEOREM 1 FOR COMPLEX $g$. We follow the proof given for real $g$, but apply it to the function $g(n)n^{-i\tau}$, employing Lemma 4 in place of Lemma 2. In this case $T=(\log x)^{1/2}$ and $T_0=K(\alpha-1)$, with $K=(\log x/\log 2w_0)^{17/19}$. This corresponds to the choice $\lambda=16/17$, $\varrho=1/19$. Note that if $\tau\neq 0$ in our present application of Lemma 4, then since $\zeta(s)=(s-1)^{-1}+O(1)$ as $s\to 1$, the upper bound $\beta$ will satisfy

$$\beta\geq\left|\frac{\zeta(\alpha+iT_0)}{\zeta(\alpha)}\right|^{1/4}\gg\left|\frac{\alpha-1}{\alpha-1+iT_0}\right|^{1/4}\gg K^{-1/4}$$

provided $x$ is sufficiently absolutely large. We so reach

$$\sum_{n\leq x/w} g(n)n^{-i\tau}=\frac{1}{w}\sum_{n\leq x} g(n)n^{-i\tau}+O\left(\frac{x}{w}\left(\frac{\log 2w_0}{\log x}\right)^{1/19}\right).$$

The proof is completed with two applications of Lemma 5, replacing $g(n)$ by $g(n)n^{-i\tau}$, $b$ by $\tau$. The parameter $L$ is estimated by applying the Cauchy-Schwarz inequality

$$L^2\leq\sum_{p\leq x}\frac{1}{p}\cdot\sum_{p\leq x}\frac{1}{p}|1-g(p)p^{-i\tau}|^2.$$

Since $|1-z|^2\leq 2(1-\operatorname{Re} z)$ when $|z|\leq 1$, and $p^{-\sigma_0}-p^{-1}\ll\log p/(p\log x)$, by part (6) of Lemma 4 the second of these prime-numbers sums does not exceed

$$2\sum_{p\leq x}\frac{1}{p^{\sigma}}(1-\operatorname{Re} g(p)p^{-i\tau})+O\left(\frac{1}{\log x}\sum_{p\leq x}\frac{\log p}{p}\right)\leq 2\log\frac{1}{\beta}+O(1),$$

and therefore

$$L^2\leq(1+o(1))\tfrac{1}{2}\log K\log\log x<\tfrac{1}{2}(\log\log x)^2$$

for all large $x$. Here we have made use of the elementary estimates

$$\sum_{p\leq x}\frac{\log p}{p}\ll\log x,\quad \sum_{p\leq x}\frac{1}{p}\leq\log\log x+O(1),\quad x\geq 2.$$

For our present purposes we may assume that $w_0\leq x^{1/2}$, otherwise Theorem 1 is trivially valid. In our application of Lemma 5

$$e^{L/2}(\log x)^{-1/2}(\log(|\tau|+\log x))^2\ll(\log x)^{(1/2\sqrt{2})-\frac{1}{2}+o(1)}\ll(\log x)^{-1/8},$$

and similarly with $x/w$ in place of $x$.

The proof of Theorem 1 is complete.

Whilst this last step seems fortuitous, restricting $K$ not to exceed a suitable power of $\log x$ will always ensure its success. Note that uniformly for $1\leq w\leq$ $\leq w_0\leq x$,

$$\sum_{n\leq x/w} g(n)=\frac{(x/w)^{i\tau}}{1+i\tau}\sum_{n\leq x/w} g(n)n^{-i\tau}+O\left(\frac{x}{w}(\log x)^{-1/8}\right),$$

and in particular

$$\frac{w}{x} \sum_{n \le x/w} g(n) \ll (1+|\tau|)^{-1} + (\log x)^{-1/8}.$$

If we cannot choose $\tau = 0$, then we may further restrict $\tau$ by $|\tau| \le (\log x)^{1/19}$.

PROOF OF THEOREM 2. Once again consider a real-valued $g$. The Dirichlet series corresponding to the function which is $g(n)$ when $(n, D) = 1$, and zero otherwise, is $\theta(s)^{-1}G(s)$, where

$$\theta(s) = \prod_{p|D} \left(1 + \sum_{k=1}^{\infty} p^{-ks} g(p^k)\right).$$

Thus

$$(\theta(s)^{-1}G(s))' = \theta(s)^{-1}G(s)' - \theta(s)^{-2}\theta(s)'G(s).$$

We form the analogue of the integral representation for $S(y)$, and as in the proof of Theorem 1, reduce the integral to a range $|\tau| \le K(\alpha - 1)$, giving

$$(7) \quad \begin{cases} Y(D) = \sum\limits_{\substack{n \le x \\ (n, D)=1}} g(n) \log n \log \dfrac{x}{n} = -\dfrac{1}{2\pi i} \displaystyle\int\limits_{\substack{(\alpha) \\ |\tau| \le K(\alpha-1)}} \theta(s)^{-1} G(s)' \dfrac{x^s ds}{s^2} \\ + \dfrac{1}{2\pi i} \displaystyle\int\limits_{\substack{(\alpha) \\ |\tau| \le K(\alpha-1)}} \dfrac{\theta'(s)G(s)x^s}{\theta(s)^2 s^2} ds + O\left(x \log x \left(\dfrac{1}{K} + \dfrac{\log \log x}{\log x}\right)^{\lambda/8}\right). \end{cases}$$

We replace $\theta(s)^{-1}$ in the first of these integrals by $\theta(1)^{-1}$. The elementary bounds

$$\sum_{p|D} \frac{\log p}{p} \ll \log \log 3D, \quad \sum_{p|D} \frac{1}{p} \le \log \log \log 3D + O(1)$$

are useful in this step. For even $D$ the extra hypothesis on the values of $g(2^k)$ ensures that provided $K(\alpha - 1)$ does not exceed a certain constant depending upon $c_1$,

$$\theta(s)^{-1} \ll \exp\left(\sum_{p|D} \frac{1}{p}\right) \ll \log \log 3D$$

holds uniformly for $|\tau| \le K(\alpha - 1)$. The replacement introduces an error $\ll xK(\log \log 3D)^2$. We may also remove the restriction $|\tau| \le K(\alpha - 1)$ from this same integral provided we introduce the factor $(1 + |\theta(1)|^{-1})$ into the error term at (7).

There is a new error term arising from the second integral at (7). For this range of $\tau$,

$$\frac{\theta'(s)}{\theta(s)} \ll 1 + \sum_{p|D} \frac{\log p}{p} \ll \log \log 3D,$$

so that the factors involving $\theta(s)$ are $\ll(\log\log 3D)^2$. An application of the Cauchy-Schwarz inequality and then Lemma 1 shows that

$$\int_{(\alpha)} |G(s)| \frac{d\tau}{|s|^2} \ll (\log x)^{1/2}.$$

Altogether

$$\frac{1}{x \log x}(Y(D)-\eta(D)Y(1)) \ll \frac{K(\log\log 3D)^2}{\log x} + \frac{(\log\log 3D)^2}{(\log x)^{1/2}} +$$

$$+\log\log 3D\left(\frac{1}{K} + \frac{\log\log x}{\log x}\right)^{\lambda/8}.$$

In this case we set $K^{\lambda/8+1} = \log x$. For large $x$ the earlier condition on $K$ is satisfied, and the error term is $\ll(\log\log 3D)^2(\log x)^{-\lambda/(8+\lambda)}$.

We remove the weights $\log n$ and $\log x/n$ from the sums $Y(D)$ and $Y(1)$ as in the treatment of the function $S$ of Theorem 1, and obtain the result stated in Theorem 2.

For complex-valued $g$ we modify the proof as we did for Theorem 1. There is one refinement. Lemma 3 is applied not to the Dirichlet series $\sum g(n)n^{-s}$ over the integers $n$ prime to $D$, but to the similar series over all the odd integers $n$. The value of $\tau$ guaranteed by that lemma will then be independent of $D$. If $D$ is odd, then the extra factor

$$1 + \sum_{k=1}^{\infty} 2^{-ks} g(2^k)$$

in $\theta(s)^{-1}G(s)$ is bounded above uniformly in the half-plane $\mathrm{Re}\,(s) \geq 1$. A factor of $\ll \log\log 3D$ is introduced into the error integrals at the outset, but does not cause any deterioration in the quality of the final error term. This completes the proof of Theorem 2.

## APPENDIX

LEMMA. *Let $A(s)=\sum_{n=1}^{\infty} a_n n^{-s}$, $B(s)=\sum_{n=1}^{\infty} b_n n^{-s}$, and suppose that $|a_n| \leq$ $\leq b_n$ holds for all positive $n$. Then*

$$\int_{(\alpha)} \left|\frac{A(s)}{s}\right|^2 d\tau \leq \int_{(\alpha)} \left|\frac{B(s)}{s}\right|^2 d\tau$$

*provided $\alpha$ is chosen so that the left-hand integral exists.*

PROOF. Let $w(z)$ denote the sum function $\sum_{n \leq z} a_n$. Formally

$$s^{-1}A(s) = \int_1^{\infty} y^{-s-1} w(y)\,dy = \int_1^{\infty} e^{-u\sigma} w(e^u) e^{-iu\tau} d\tau,$$

so that $(s\sqrt{2\pi})^{-1}A(s)$ as a function of $\tau$, and $e^{-u\sigma}w(e^u)$ as a function of $u$, are Fourier transforms. Then by Plancherel's theorem

$$\frac{1}{2\pi}\int_{(\alpha)}\left|\frac{A(s)}{s}\right|^2 d\tau=\int_0^\infty e^{-2u\sigma}|w(e^u)|^2 du,$$

so long as the left-hand integral exists in the usual Lebesgue sense. By hypothesis.

$$|w(e^u)|\leq \sum_{n\leq e^u} b_n,$$

and we may reverse the steps of the argument to complete the proof.

REMARKS. This argument shows at once that the Riemann zeta function is the extremal for the functional

$$G\longmapsto \int_{(\alpha)}\left|\frac{G(s)}{s}\right|^2 d\tau,\quad \alpha>1.$$

It is also effectively an extremal for the functionals defined by replacing the function $G$ in this integrand with $-G'/G$, or $|G|^\beta$ for some $\beta>0$. For example, if $\sigma>1$, then by considering Euler products, as in Elliott [4], Chapter 6, Lemma (6.6), pp. 230–231, we have

$$|G(s)|^\beta \ll \left|\exp\left(\frac{\beta}{2}\sum_{p\geq 3}\frac{g(p)}{p^s}\right)\right|^2.$$

By the above remark

$$\int_{(\alpha)}\frac{|G(s)|^\beta}{|s|^2}d\tau \ll \int_{(\alpha)}\frac{|\zeta(s)|^\beta}{|s|^2}d\tau,$$

since a further appeal to Euler products shows that

$$\left|\exp\left(\frac{\beta}{2}\sum_{p\geq 3}\frac{1}{p^s}\right)\right| \ll |\zeta(s)|^\beta.$$

For $\beta>1$ we may continue

$$\int_{(\alpha)}\frac{|\zeta(s)|^\beta}{|s|^2}d\tau \ll \int_{(\alpha)}\left(\frac{1}{|s-1|}+\log(2+|s|)\right)^\beta \frac{d\tau}{|s|^2} \ll (\alpha-1)^{1-\beta},$$

to obtain the second of the bounds asserted in Lemma 1; and so on.

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